

Faster DTMF Decoding

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Abstract. In this paper, a new method for the decoding of DTMF signals is proposed. The approach, which applies the Arithmetic Fourier Transform, is more efficient, in terms of computational complexity, than existing techniques. Theoretical aspects and features that determine the accuracy and the complexity of the proposed method are discussed.

1 Introduction

Computational complexity is a decisive figure of merit in algorithms intended for frequency analysis. Bruns [1] developed a method for computing the coefficients of a Fourier series using the Möbius inversion formula for finite series. The technique, later called the Arithmetic Fourier Transform (AFT) [2], requires mainly trivial multiplications, except for a few scaling factors. Tufts and Sadasiv [3] discovered a very similar algorithm that had the constraint to only deal with even signals. This constraint was later removed by Reed and Tufts [4]. Reed and Shih [5] improved the previous algorithm and proposed the simplified AFT. In this paper, a new method for decoding DTMF (Dual-Tone Multi-Frequency) signals is proposed, which is based on the AFT. The method applies the simplified AFT that has a lower computational complexity than its previous versions. Specifically, the number of multiplications involved in the decoding operation is much lower than that required by an FFT or by the Goertzel algorithm [6]. In the next section basic facts about the AFT are presented. On section 3, the relation between the DFT and the Fourier series is shown. The choice of AFT parameters, such as the sampling rate and transform length, are discussed in section 4. Some numerical results concerning the decoding errors are shown. Section 5 presents the conclusions of the paper.

2 The Arithmetic Fourier Transform

This section briefly reviews a few basic facts concerning the AFT, algorithm that is in the heart of the proposed DTMF decoding method.

Theorem 1 (The Möbius inversion formula for finite series [4]): Assume that n is a positive integer and f_n is a nonzero sequence confined to the interval $1 \leq n \leq N$.

If $g_n = \sum_{k=1}^{\lfloor N/n \rfloor} f_{kn}$, then $f_n = \sum_{m=1}^{\lfloor N/n \rfloor} \mu(m)g_{mn}$, where μ is the Möbius function.

2.1 Reed-Tufts

Let $v(t)$ be a real signal with period T , whose finite Fourier series has the form

$$v(t) = a_0 + \sum_{k=1}^N a_k \cos\left(\frac{2\pi k t}{T}\right) + \sum_{k=1}^N b_k \sin\left(\frac{2\pi k t}{T}\right), \quad (1)$$

where $f_0=1/T$ and a_0 is the mean of $v(t)$. If the dc component is removed from $v(t)$, one obtains $\bar{v}(t)$. Shifting the periodic function $\bar{v}(t)$ by an amount αT , where $|\alpha| < 1$,

yields $\bar{v}(t + \alpha T) = \sum_{k=1}^N c_k(\alpha) \cos\left(\frac{2\pi k t}{T}\right) + \sum_{k=1}^N d_k(\alpha) \sin\left(\frac{2\pi k t}{T}\right)$, where

$$c_k(\alpha) = a_k \cos(2\pi k \alpha) + b_k \sin(2\pi k \alpha), \quad d_k(\alpha) = -a_k \sin(2\pi k \alpha) + b_k \cos(2\pi k \alpha). \quad (2)$$

Definition 1: The k th partial average is

$$S_k(\alpha) = \frac{1}{k} \sum_{m=0}^{k-1} \bar{v}\left(\frac{m}{k}T + \alpha T\right), \quad \text{where } -1 < \alpha < 1. \quad (3)$$

a_k and b_k can be expressed as a function of $c_k(\alpha)$. Firstly, we relate $c_k(\alpha)$ with the S_k .

Theorem 2: The coefficients $c_k(\alpha)$ can be computed by theorem 1

$$c_k(\alpha) = \sum_{l=1}^{\lfloor N/k \rfloor} \mu(l) S_{lk}(\alpha). \quad (4)$$

The coefficients a_k and b_k for $k=2^r(2m+1)$ are computed by

$$a_k = c_k(0), \quad b_k = (-1)^m c_k\left(\frac{1}{2^{r+2}}\right), \quad k = 1, \dots, N \quad (5)$$

where r and m are obtained from the factorization of k .

2.2 Reed-Shih (Simplified AFT)

In this method, the partial averages are redefined according to Bruns [1].

Definition 2 (Bruns averages): The Bruns partial averages, $B_{2k}(\alpha)$, are defined by

$$B_{2k}(\alpha) = \frac{1}{2k} \sum_{m=0}^{2k-1} (-1)^m v\left(m \frac{T}{2k} + \alpha T\right). \quad (6)$$

From the definition of c_k in (2) and using theorem 2 and definition 1, we obtain [5].

Theorem 3: The coefficients $c_k(\alpha)$ are given by the Möbius inversion formula:

$$c_k(\alpha) = \sum_{l=1,3,\dots}^{\lfloor N/k \rfloor} \mu(l) B_{2kl}(\alpha). \quad (7)$$

In (2), two conditions can be distinguished: $a_k=c_k(0)$ and $b_k=c_k(1/4k)$. From them and theorem 3, the next result follows.

Theorem 4 (Reed-Shih): The Fourier coefficients a_k and b_k , $k=1,\dots,N$, are given by

$$a_0 = \frac{1}{T} \int_0^T v(t) dt, \quad a_k = \sum_{l=1,3,\dots}^{\lfloor N/k \rfloor} \mu(l) B_{2kl}(0), \quad b_k = \sum_{l=1,3,\dots}^{\lfloor N/k \rfloor} \mu(l) (-1)^{\frac{l-1}{2}} B_{2kl}(1/4kl). \quad (8)$$

3 The DFT and the Coefficients of the Fourier Series

This section shows the relation between the DFT of a sequence and the Fourier coefficients of the continuous version of the signal. This allows to take advantage of the low computational complexity of the AFT for implementing the decoder.

Definition 3 (The Discrete Fourier Transform): Let v be a complex-valued N -dimensional vector. The DFT of v is the vector V , of elements $V[k]$, given by

$$V[k] = \sum_{n=0}^{N-1} v[n] \exp\left(-j \frac{2\pi kn}{N}\right), \quad k = 0, 1, \dots, N-1. \quad (9)$$

The inverse transform is given by

$$v[n] = \sum_{k=0}^{N-1} V[k] \exp\left(j \frac{2\pi kn}{N}\right), \quad n = 0, 1, \dots, N-1. \quad (10)$$

The Fourier series up to the $(N/2)$ -th harmonic of $v(t)$ is

$$v(t) = a_0 + \sum_{k=1}^{N/2} a_k \cos\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{N/2} b_k \sin\left(\frac{2\pi kt}{T}\right), \quad (11)$$

where N is even and all the other terms of the series are supposed to be negligible. By sampling N equidistant points through a period of $v(t)$, a sequence $v[n]$ is obtained. Then, the discrete version of (11) can be written as

$$v[n] = a_0 + a_{N/2} (-1)^n + \sum_{k=1}^{(N-2)/2} a_k \cos\left(\frac{2\pi nk}{N}\right) + \sum_{k=1}^{(N-2)/2} b_k \sin\left(\frac{2\pi nk}{N}\right). \quad (12)$$

Writing the components $V[k]$ of the DFT in cartesian form $V[k] = \text{Re}\{V[k]\} + j \text{Im}\{V[k]\}$ and substituting in (10), leads to

$$v[n] = \frac{V[0]}{N} + \frac{V[N/2](-1)^n}{N} + \frac{2}{N} \sum_{k=0}^{N/2} \text{Re}\{V[k]\} \cos\left(\frac{2\pi kn}{N}\right) - \frac{2}{N} \sum_{k=0}^{N/2} \text{Im}\{V[k]\} \sin\left(\frac{2\pi kn}{N}\right). \quad (13)$$

Comparing expressions (12) and (13), we may write $k=1, \dots, (N-2)/2$.

$$a_0 = \frac{V[0]}{N}, \quad a_{N/2} = \frac{V[N/2]}{N}, \quad a_k = \frac{2 \operatorname{Re}\{V[k]\}}{N}, \quad b_k = -\frac{2 \operatorname{Im}\{V[k]\}}{N}, \quad (14)$$

If N is odd,

$$a_0 = \frac{V[0]}{N}, \quad a_k = \frac{2 \operatorname{Re}\{V[k]\}}{N}, \quad b_k = -\frac{2 \operatorname{Im}\{V[k]\}}{N}, \quad \text{for } k=1, \dots, \frac{(N-1)}{2}, \quad (15)$$

$$\sqrt{a_k^2 + b_k^2} = \frac{2}{N} |V[k]|. \quad (16)$$

4 DTMF Decoding via AFT

In the DTMF system, each key-press generates the sum of two audible tones. When a DTMF signal is received, its frequency content is analysed to identify which digit was transmitted. This analysis can be made by computing its DFT. The magnitudes of the eight components associated to the frequencies nearest the DTMF frequencies are observed. Then, decoding is accomplished by selecting the two higher components. Typically, either a radix-2 FFT algorithm is employed, or the Goertzel algorithm is used when computing only a few components of the DFT.

4.1 The Sampling Frequency and the Length of the Transform

Spectrum analysis of a discrete-time signal by the AFT requires a choice of parameters that are relevant to the accuracy and computational efficiency of the process. The sampling rate of the original continuous signal is one such factor. We set this frequency to 8 kHz, a standard value of PCM-based telephone systems. Another parameter is the transform length N . A suitable value for N , one that allows the detection of the eight DTMF frequencies, should be selected. The relation

$$k = f_k N / f_s, \quad (17)$$

where f_s is the sampling frequency, gives the index k of the coefficient that corresponds to the detected frequency f_k . In general, it returns a noninteger value for k , which is rounded. It is thus possible to define a relative error measure by

$$E_{R,k} = |f_k - \tilde{f}| / f_k, \quad (18)$$

where \tilde{f} is the rounded frequency. We calculate the mean relative error, \bar{E}_R , by averaging out $E_{R,k}$ over the DTMF frequencies. Higher quality detection can be achieved by using the transform length that minimises \bar{E}_R . An exhaustive search procedure leads to $N=114$.

4.2 Applying the Arithmetic Fourier Transform

Equation (17) provides the coefficient indexes necessary to decoding. Equation (8) shows the Bruns averages needed to compute such coefficients (table 1). From (6), values of the signal for fractional times must be known, what requires interpolation. The type of interpolation affects on the error and the cost of the algorithm. The decoding was implemented using linear interpolation. The coefficients of the indexes k shown in table 2, were computed for each of the 16 DTMF signals.

Table 1. DTMF frequencies, indexes of coefficients and Bruns averages, $f_s = 8$ kHz, $N=114$.

$f(\text{Hz})$	k	Bruns averages	$f(\text{Hz})$	k	Bruns averages
697	10	$B_{20}, B_{60}, B_{100}, B_{140}, B_{220}$	1209	17	B_{34}, B_{102}, B_{170}
770	11	$B_{22}, B_{66}, B_{110}, B_{154}$	1336	19	B_{38}, B_{114}, B_{190}
852	12	$B_{24}, B_{72}, B_{120}, B_{168}$	1477	21	B_{42}, B_{126}, B_{210}
941	13	$B_{26}, B_{78}, B_{130}, B_{182}$	1633	23	B_{46}, B_{138}

The key point of adopting an AFT-DTMF decoding is its low complexity. The number of floating-point multiplications and the number of additions was computed following [5]. Table 2 presents a comparison between the complexity of the proposed method and Goertzel algorithm [6].

Table 2. Computational Complexity: Simplified AFT, Goertzel algorithm, $f_s = 8$ kHz, $N= 114$.

Operation	Simplified AFT	Goertzel
Multiplication	56	904
Addition	~ 4500	1800

The higher number of additions needed in the simplified AFT is fully compensated for the difference in the number of multiplications. A similar result is observed when comparing the AFT with the Cooley-Tukey FFT. For $N=128$, the FFT requires 712 multiplications and 2504 additions [6].

4.3 An Error Measurement

The interpolations required to compute Bruns averages are responsible for errors in the estimates of a_k and b_k . In order to evaluate this inaccuracy, we use equations (14) and (15) to estimate the tones that identify a digit in the DTMF keypad. For instance, for the signal that represents the digit “1”, we estimate $V[10]$ and $V[17]$. The errors in these components are computed, relatively to their exact (DFT) values:

$$E_k = \left| \frac{|V[k]| - |\tilde{V}[k]|}{|V[k]|} \right|, \quad (19)$$

where $|\tilde{V}[k]|$ is the estimate of the DFT component of index k , and $|V[k]|$ is its exact value. After computing E_k for each component, we cluster those that correspond to a same frequency and calculate the arithmetic mean, \bar{E}_k . Table 3 shows DTMF

frequencies, and the associated estimation mean error. The error presented in table 3 significantly varies with the sampling frequency and transform length. The same happens with the complexity of the algorithm. It is thus possible to vary f_s and N , establishing a trade-off between these parameters so as to suit specific project needs.

Table 3. DTMF frequencies, coefficient indexes and estimation mean errors, $f_s=8$ kHz, $N=114$.

f (Hz)	697	770	852	941	1209	1336	1477	1633
K	10	11	12	13	17	19	21	23
$\bar{E}_{Rn} \times 10^{-2}$	3.86	6.48	4.19	10.82	4.51	7.30	8.13	5.87

5 Conclusions

This paper offered a new tool for decoding DTMF signals: the Arithmetic Fourier Transform. Some theoretical and practical aspects of the method were discussed, and results concerning the algorithm computational complexity were presented. The AFT decoding presents a low number of floating-point multiplications when compared to existing algorithms for the same application, allowing a faster DTMF decoding. Details for further simplifying the decoding algorithm are currently under investigation, such as, for instance, to deal only with the most significant Bruns partial averages to compute a Fourier coefficient.

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