Elliptic-Cylindrical Wavelets: The Mathieu Wavelets

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***Abstract*—This note introduces a new family of wavelets and a multiresolution analysis that exploits the relationship between analyzing filters and Floquet's solution of Mathieu differential equations. The transfer function of both the detail and the smoothing filter is related to the solution of a Mathieu equation of the odd characteristic exponent. The number of notches of these filters can be easily designed. Wavelets derived by this method have potential application in the fields of optics and electromagnetism.**

***Index Terms*—Floquet's theorem, Mathieu equation, waveguides, wavelets.**

I. Introduction

In 1868, THE FRENCH mathematician É. L. Mathieu introduced in his "*Memoir on Vibrations of an Elliptic Membrane*" a family of differential equations that are nowadays termed Mathieu equations [1]. Mathieu's equation is related to the wave equation for the elliptic cylinder. Mathieu is notably remembered for his discovery of sporadic simple groups [2]. This letter is particularly concerned with the canonical form of the Mathieu equation. For $a \in \mathbb{R}$, $q \in \mathbb{C}$, the Mathieu equation is given by

$$\frac{d^2y}{d\omega^2} + (a - 2q\cos(2\omega))y = 0. \tag{1}$$

The Mathieu equation is a linear second-order differential equation with periodic coefficients. This equation was shown later to be also related to quantum mechanicals; the parameters a and q denote the energy level and an intensity, respectively. For $q = 0$, it reduces to the well-known harmonic oscillator, a being the square of the frequency [3]. The solution of (1) is the elliptic-cylindrical harmonic, known as Mathieu functions. In addition to being theoretically fascinating, Mathieu functions are applicable to a wide variety of physical phenomena, e.g., diffraction, amplitude distortion, inverted pendulum, stability of a floating body, radio frequency quadrupole, and vibration in a medium with modulated density [4]. They have also long been applied on a broad scope of waveguide problems involving elliptical geometry, including the following:

1) analysis for weak guiding for step index elliptical core optical fibers [5];
2) power transport of elliptical waveguides [6], [7];
3) evaluating radiated waves of elliptical horn antennas [8];
4) elliptical annular microstrip antennas with arbitrary eccentricity [9];
5) scattering by a coated strip [10].

The aim of this letter is to propose a new family of wavelets based on Mathieu differential equations. Wavelets are a well-known tool for differential equation solving [11]–[13]. However, in this work, we show another connection between wavelets and differential equations: the design of new wavelets from the solution of a differential equation.

II. Mathieu Equations

In general, the solutions of (1) are not periodic. However, for a given q, periodic solutions exist for infinitely many special values (eigenvalues) of a. For several physically relevant solutions, y must be periodic of period π or 2π. It is also convenient to distinguish even and odd periodic solutions, which are termed Mathieu functions of the first kind. One of four simpler types can be considered: periodic solution (π or 2π) symmetry (even or odd). For $q \neq 0$, the only periodic solution y corresponding to any characteristic value $a = a_r(q)$ or $a = b_r(q)$ has the following notation.

Even periodic solution

$$\mathrm{ce}_r(\omega, q) = \sum_m A_{r,m} \cos m\omega \qquad \text{for } a = a_r(q) \tag{2a}$$

Odd periodic solution

$$\mathrm{se}_r(\omega, q) = \sum_m A_{r,m} \sin m\omega \qquad \text{for } a = b_r(q) \tag{2b}$$

where the sums are taken over even (respectively odd) values of m if the period of y is π (respectively 2π). Given r, henceforth, we denote $A_{r,m}$ by A_m for short. Elliptic cosine and elliptic sine functions are represented by ce and se, respectively. Interesting relationships are found when $q \to 0$, $r \neq 0$ [14]

$$\lim_{q\to 0} \mathrm{ce}_r(\omega, q) = \cos(r\omega) \quad \lim_{q\to 0} \mathrm{se}_r(\omega, q) = \sin(r\omega). \tag{3}$$

One of the most powerful results of Mathieu's functions is Floquet's theorem [15]. It states that periodic solutions of (1) for any pair (a, q) can be expressed in either of the forms

$$\begin{aligned} y(\omega) &= F_\nu(\omega) = e^{j\nu\omega}P(\omega) \\ y(\omega) &= F_\nu(-\omega) = e^{-j\nu\omega}P(-\omega) \end{aligned} \tag{4}$$

where ν is a constant depending on a and q, and $P(\cdot)$ is π-periodic in ω. The constant ν is called the characteristic exponent. If ν is an integer, then $F_\nu(\omega)$ and $F_\nu(-\omega)$ are linear dependent solutions. Furthermore, $y(\omega + k\pi) = e^{j\nu k\pi}y(\omega)$ or $y(\omega + k\pi) =$

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$e^{-j\nu k\pi}y(\omega)$, for the solution $F_\nu(\omega)$ or $F_\nu(-\omega)$, respectively. We assume that the pair (a, q) is such that $|\cosh(j\nu\pi)| < 1$ so that the solution $y(\omega)$ is bounded on the real axis [16]. The general solution of Mathieu's equation ($q \in \mathbb{R}$, ν noninteger) has the form

$$y(\omega) = c_1 e^{j\nu\omega} P(\omega) + c_2 e^{-j\nu\omega} P(-\omega) \tag{5}$$

where c_1 and c_2 are arbitrary constants.

All bounded solutions—those of fractional as well as integral order—are described by an infinite series of harmonic oscillations whose amplitudes decrease with increasing frequency. In the wavelet framework we are basically concerned with even solutions of period 2π. In such cases, there exist recurrence relations among the coefficients [14]

$$\begin{aligned} (a-1-q)A_1 - qA_3 &= 0 \\ (a-m^2)A_m - q(A_{m-2} + A_{m+2}) &= 0 \\ m \geq 3, \quad m \text{ odd.} \end{aligned} \tag{6}$$

In the sequel, wavelets are denoted by $\psi(t)$ and scaling functions by $\phi(t)$, with corresponding spectra $\Psi(\omega)$ and $\Phi(\omega)$, respectively.

III. Mathieu Wavelets

Wavelet analysis has matured rapidly over the past years and has been proved to be invaluable for scientists and engineers [17]. Wavelet transforms have lately gained extensive applications in an amazing number of areas.[1] The equation $\phi(t) = \sqrt{2}\sum_{n\in\mathbb{Z}} h_n \phi(2t-n)$, which is known as the *dilation* or *refinement equation*, is the chief relation determining a multiresolution analysis (MRA) [18].

A. Two-Scale Relation of Scaling Function and Wavelet

Defining the spectrum of the smoothing filter $\{h_k\}$ by $H(\omega) \triangleq 1/\sqrt{2}\sum_{k\in\mathbb{Z}} h_k e^{-j\omega k}$, the central equations (in the frequency domain) of an MRA are [19]

$$\Phi(\omega) = H\left(\frac{\omega}{2}\right)\Phi\left(\frac{\omega}{2}\right) \quad \Psi(\omega) = G\left(\frac{\omega}{2}\right)\Phi\left(\frac{\omega}{2}\right) \tag{7}$$

where $G(\omega) \triangleq 1/\sqrt{2}\sum_{k\in\mathbb{Z}} g_k e^{-j\omega k}$ is the transfer function of the detail filter.

The orthogonality condition corresponds to [19]

$$H(0) = 1 \quad H(\pi) = 0 \tag{8a}$$

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1 \tag{8b}$$

$$H(\omega) = -e^{-j\omega} G^*(\omega+\pi). \tag{8c}$$

B. Filters of a Mathieu MRA

The subtle liaison between Mathieu's theory and wavelets was found by observing that the classical relationship

$$\Psi(\omega) = e^{-j\omega/2} H^*\left(\frac{\omega}{2} - \pi\right)\Phi\left(\frac{\omega}{2}\right) \tag{9}$$

presents a remarkable similarity to a Floquet's solution of a Mathieu's equation, since $H(\omega)$ is a periodic function.

As a first attempt, the relationship between the wavelet spectrum and the scaling function was put in the form

$$\frac{\Psi(\omega)}{\Phi\left(\frac{\omega}{2}\right)} = e^{-j\omega/2} H^*\left(\frac{\omega}{2} - \pi\right). \tag{10}$$

Here, on the second member, neither ν is an integer, nor $H(\cdot)$ has a period π. By an appropriate scaling of this equation, we can rewrite it as

$$\frac{\Psi(4\omega)}{\Phi(2\omega)} = e^{-j2\omega} H^*(2\omega - \pi). \tag{11}$$

Defining a new function $Y(\omega) \triangleq \Psi(4\omega)/\Phi(2\omega)$, we recognize that it has a nice interpretation in the wavelet framework. First, we recall that $\Psi(\omega) = G(\omega/2)\Phi(\omega/2)$ so that $\Psi(2\omega) = G(\omega)\Phi(\omega)$. Therefore, the function related to Mathieu's equation is exactly $Y(\omega) = G(2\omega)$. Introducing a new variable z, which is defined according to $2z \triangleq 2\omega - \pi$, it follows that $-Y(z + \pi/2) = e^{-j2z} H^*(2z)$. The characteristic exponent can be adjusted to a particular value ν

$$-e^{-j(\nu-2)z} Y\left(z + \frac{\pi}{2}\right) = e^{-j\nu z} H^*(2z). \tag{12}$$

Defining now $P(-z) \triangleq H^*(2z) = \sum_{k\in\mathbb{Z}} c_{2k} e^{jz2k}$, where $c_{2k} \triangleq 1/\sqrt{2}h_k^*$, we figure out that the right side of the above equation represents a Floquet's solution of some differential Mathieu equation. The function $P(\cdot)$ is π-periodic, verifying the initial condition $P(0) = 1/\sqrt{2}\sum_k h_k = 1$, as expected. The filter coefficients are all assumed to be real. Therefore, there exist a set of parameters (a_G, q_G) such that the auxiliary function

$$y_\nu(z) \triangleq -e^{-j(\nu-2)z} Y_\nu\left(z + \frac{\pi}{2}\right) \tag{13}$$

is a solution of the following Mathieu equation:

$$\frac{d^2 y_\nu}{dz^2} + (a_G - 2q_G\cos(2z))y_\nu = 0 \tag{14}$$

subject to $y_\nu(0) = -Y(\pi/2) = -G(\pi) = -1$ and $\cos(\pi\nu) - y_\nu(\pi) = 0$, i.e, $y_\nu(\pi) = (-1)^\nu$.

In order to investigate a suitable solution of (14), boundary conditions are established for predetermined a, q. It turns out that when ν is zero or an integer, a belongs to the set of characteristic values $a_r(q)$. Furthermore, $\nu = r$ is associated with $a_r(q)$. The even (2π-periodic) solution of such an equation is given by

$$y_\nu(z) = -\frac{\mathrm{ce}_\nu(z,q)}{\mathrm{ce}_\nu(0,q)}. \tag{15}$$

The $Y_\nu(\omega)$ function associated to $y_\nu(z)$ and related to the detail filter of a "Mathieu MRA" is thus

$$Y_\nu(\omega) = G_\nu(2\omega) = e^{j(\nu-2)(\omega-\pi/2)}\frac{\mathrm{ce}_\nu\left(\omega - \frac{\pi}{2}, q\right)}{\mathrm{ce}_\nu(0,q)}. \tag{16}$$

[1] A. Fournier, Wavelets and their applications in computer graphics. Course notes from *Proc. 1995 ACM Conf. Computer Graphics (SIGGRAPH '95)*. ftp://ftp.cs.ucb.ca/pub/local/bobl/wvlt.

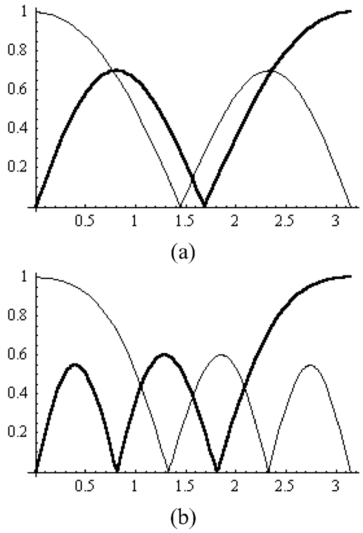

Fig. 1. Magnitude of the transfer function for Mathieu multiresolution analysis filters. (Solid line) Smoothing filter $|H_\nu(\omega)|$ and (bold line) detail filter $|G_\nu(\omega)|$ for a few Mathieu parameters. (a) $\nu = 3$, $q = 3$, $a = 9.915\,506\,290\,452\,134$. (b) $\nu = 5$, $q = 15$, $a = 31.957\,821\,252\,172\,874$.

Finally, the transfer function of the detail filter of a Mathieu wavelet is

$$G_\nu(\omega) = e^{j(\nu-2)(\omega-\pi/2)} \frac{\mathrm{ce}_\nu\left(\frac{\omega-\pi}{2}, q\right)}{\mathrm{ce}_\nu(0,q)}. \tag{17}$$

The characteristic exponent ν should be chosen so as to guarantee suitable initial conditions, i.e., $G_\nu(0) = 0$ and $G_\nu(\pi) = 1$, which are compatible with wavelet filter requirements. Therefore, ν must be odd. It is interesting to remark that the magnitude of the above transfer function corresponds exactly to the modulus of a elliptic sine [16]

$$|G_\nu(\omega)| = \left| \frac{\mathrm{se}_\nu\left(\frac{\omega}{2}, -q\right)}{\mathrm{ce}_\nu(0,q)} \right|. \tag{18}$$

The solution for the smoothing filter $H(\cdot)$ can be found out via quadrature mirror filter bank conditions [18], yielding

$$H_\nu(\omega) = -e^{-j\nu\omega/2} \frac{\mathrm{ce}_\nu\left(\frac{\omega}{2}, q\right)}{\mathrm{ce}_\nu(0,q)}. \tag{19}$$

In this case, we find $H_\nu(\pi) = 0$ and

$$|H_\nu(\omega)| = \left| \frac{\mathrm{ce}_\nu\left(\frac{\omega}{2}, q\right)}{\mathrm{ce}_\nu(0,q)} \right|. \tag{20}$$

Given q, the even first-kind Mathieu function with characteristic exponent ν is given by $\mathrm{ce}_\nu(\omega,q) = \sum_{l=0}^{\infty} A_{2l+1} \cos(2l+1)\omega$, in which $\mathrm{ce}_\nu(0,q) = \sum_{l=0}^{\infty} A_{2l+1}$. The G and H filter coefficients of a Mathieu MRA can be expressed in terms of the values $\{A_{2l+1}\}_{l\in\mathbb{Z}}$ of the Mathieu function as

$$\frac{h_l^\nu}{\sqrt{2}} = -\frac{\frac{A_{|2l-\nu|}}{2}}{\mathrm{ce}_\nu(0,q)} \quad \frac{g_l^\nu}{\sqrt{2}} = (-1)^l \frac{\frac{A_{|2l+\nu-2|}}{2}}{\mathrm{ce}_\nu(0,q)}. \tag{21}$$

It is straightforward to show that $h_{-l}^\nu = h_{l+\nu}^\nu$, $\forall\, l > 0$. The normalizing conditions are $1/\sqrt{2}\sum_{k=-\infty}^{\infty} h_k^\nu = -1$ and $\sum_{k=-\infty}^{\infty}(-1)^k h_k^\nu = 0$.

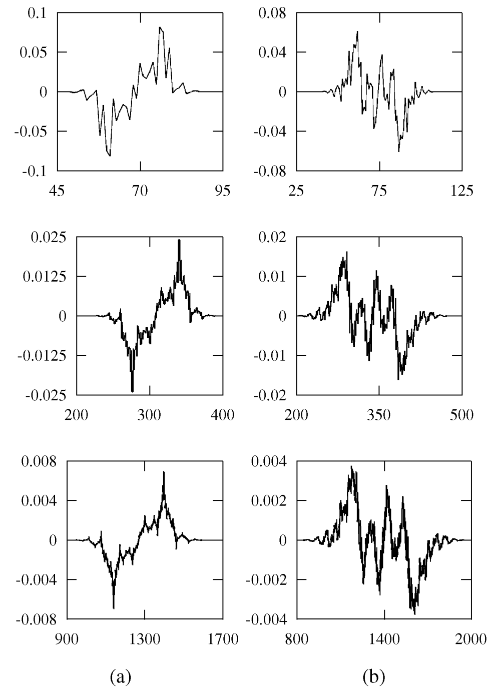

Fig. 2. FIR-based approximation of Mathieu wavelets as the number of iteration increases (two, four, and six iterations, respectively). Filter coefficients holding $|h| < 10^{-10}$ were thrown away (19 retained coefficients per filter in both cases). (a) Mathieu wavelet with $\nu = 3$ and $q = 3$ and (b) Mathieu wavelet with $\nu = 5$ and $q = 15$.

IV. Examples

Illustrative examples of filter transfer functions for a Mathieu MRA are shown in Fig. 1, for $\nu = 3$ and 5, and a particular value of q (numerical solution obtained by five-order Runge–Kutta method). The value of a is adjusted to an eigenvalue in each case, leading to a periodic solution. Such solutions present a number of ν zeroes in the interval $|\omega| < \pi$. We observe lowpass behavior (for the filter H) and highpass behavior (for the filter G), as expected. Mathieu wavelets can be derived from the lowpass reconstruction filter by the cascade algorithm. Infinite-impulse response filters (IIRs) should be applied, since the Mathieu wavelet has no compact support. However, a finite-impulse response (FIR) approximation can be generated by discarding negligible filter coefficients, say less than 10^{-10}. In Fig. 2, an emerging pattern that progressively looks like the wavelet shape is shown for some couple of parameters a and q. Waveforms were derived using the Matlab wavelet toolbox. As with many wavelets, there is no nice analytical formula for describing Mathieu wavelets.

V. Conclusion

A new and wide family of elliptic-cylindrical wavelets was introduced. It was shown that the transfer functions of the corresponding multiresolution filters are related to Mathieu equation solutions. The magnitude of the detail and smoothing filters corresponds to first-kind Mathieu functions with an odd characteristic exponent. The number of zeroes of the highpass $|G(\omega)|$ and lowpass $|H(\omega)|$ filters within the interval $|\omega| < \pi$ can be appropriately designed by choosing the characteristic exponent. This seems to be the first connection found between Mathieu equations and wavelet theory. It opens new perspectives on linking wavelets and solutions of other differential equations (e.g., associated Legendre functions).

Although there exist plenty of potential applications for Mathieu wavelets, none are presented: we just disseminate the major ideas, letting further research be investigated. For instance, this new family of wavelets could be an interesting tool for analyzing optical fibers due to its "elliptical" symmetry. They could as well be beneficial when examining molecular dynamics of charged particles in electromagnetic traps such as Paul trap or the mirror trap for neutral particles [20], [21].

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